4544. *Proposed by Burghelea Zaharia.*

Calculate

$$\int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right)\frac{dx}{x}.$$

We received 17 correct, 1 incomplete, and 1 incorrect solutions. The majority of the solutions used transformations using the substitution method for definite integrals and virtuoso elimination of non-elementary integrals. There were also some solutions which used power series and their definite integrals where Euler's answer for the Basel Problem was helpful. Some of the mistakes in the solutions *could have been avoided if the solvers checked their answer using, for example, software integral calculators. W. Janous and S. Jason generalized the problem by proving that*

$$\int_1^a \ln\left(\frac{x^4+a^2}{x^2+a^2}\right)\frac{dx}{x} = \frac{1}{2}(\ln a)^2.$$

Here we present the solution by the Missouri State University Problem Solving Group.

In the integral, let $x = 2y$, then

$$\begin{aligned} I &= \int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right)\frac{dx}{x} \qquad (1)\\ &= \int_{\frac{1}{2}}^1 \ln\left(\frac{4y^4+1}{y^2+1}\right)\frac{dy}{y}\\ &= \int_{\frac{1}{2}}^1 \ln(1+4y^4)\frac{1}{y}dy - \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy. \end{aligned}$$

In the first integral, let $z = 2y^2$, or $y = \sqrt{z}/\sqrt{2}$, then

$$\begin{aligned} I &= \int_{\frac{1}{2}}^2 \ln(1+z^2)\frac{1}{2z}dz - \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy\\ &= \frac{1}{2}\int_1^2 \ln(1+z^2)\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy\\ &= \frac{1}{2}\int_1^2 (\ln z^2 + \ln(1+z^{-2}))\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz\\ &= \int_1^2 \frac{\ln z}{z}dz + \frac{1}{2}\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz\\ &= \frac{1}{2}(\ln 2)^2 + \frac{1}{2}\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz - \frac{1}{2}\int_{\frac{1}{2}}^1 \ln(1+z^2)\frac{1}{z}dz. \qquad (2) \end{aligned}$$

However, let $y = 1/z$, we will have

$$\int_1^2 \ln(1+z^{-2})\frac{1}{z}dz = \int_1^{\frac{1}{2}} \ln(1+y^2)y(-y^{-2})dy = \int_{\frac{1}{2}}^1 \ln(1+y^2)\frac{1}{y}dy.$$

Hence, the second and the third terms in Eq. (1) add to zero.

As a result, we have

$$I = \int_1^2 \ln\left(\frac{x^4+4}{x^2+4}\right)\frac{dx}{x} = \frac{1}{2}(\ln 2)^2.$$